Hilbert's Tenth Problem

Yuri V. Matiyasevich

foreword by Martin Davis
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Theoretical computer science has now undergone several decades of development. The “classical” topics of automata theory, formal languages, and computational complexity have become firmly established, and their importance to other theoretical work and to practice is widely recognized. Stimulated by technological advances, theoreticians have been rapidly expanding the areas under study, and the time delay between theoretical progress and its practical impact has been decreasing dramatically. Much publicity has been given recently to breakthroughs in cryptography and linear programming, and steady progress is being made on programming language semantics, computational geometry, and efficient data structures. Newer, more speculative, areas of study include relational databases, VLSI theory, and parallel and distributed computation. As this list of topics continues expanding, it is becoming increasingly important that the most significant work be distilled and communicated in a manner that will facilitate further research and application of this work. By publishing comprehensive books and specialized monographs on the theoretical aspects of computer science, the series on Foundations of Computing provides a forum in which important research topics can be presented in their entirety and placed in perspective for researchers, students, and practitioners alike.

Michael R. Garey
Albert R. Meyer
A Note on the Translation

The English version of this book is a translation from the Russian, but without an official translator. A complete English first draft was prepared by the author, Yuri Matiyasevich, himself. The task of transforming this draft into a reasonably polished English text was undertaken by David Jones of MIT Press and myself, neither of us knowing any Russian. David began by translating Yuri's computer files into \TeX, specifically \texttt{AMS-LaTeX}, and by helping me to set up this dialect of \TeX in my computer. As David completed each chapter, he passed it on to me. After I had made a first pass it would go to David and to Yuri for multiple iterations of the process. In the process, Yuri also added numerous items to the bibliography that were not present in the Russian edition.

David and I were in almost daily contact by electronic mail for close to a year, joined by Yuri when the communication channels permitted. Although David and I have never met, I think we know one another rather well by now. Many were the arguments we had over the need for a comma, and never again will I write the word "which" without wondering whether David would permit me to use it rather than "that" in the given context.

The Russian name of the discipline referred to in English as recursion theory or computability theory translates literally as the theory of algorithms. In current usage however, that English phrase suggests a rather different subject, namely algorithmic analysis. For this reason, we decided to use the English terminology. In all other respects, we have tried to stay as close to the original as possible.

Throughout the book, we have transliterated Cyrillic text according to the current conventions of the AMS Mathematical Reviews. The only exceptions are names of people who have a clear preference for an alternative spelling.

Martin Davis
Foreword

While I was still an undergraduate at City College in New York, I read my teacher E. L. Post's plaint that Hilbert's Tenth Problem "begs for an unsolvability proof." This was the beginning of my lifelong obsession with the problem. Although I have had the good fortune to be able to make some contributions towards the "unsolvability proof" for which the problem was begging, my greatest insight turned out to be a thought I had uttered in jest. During the 1960s I often had occasion to lecture on Hilbert's Tenth Problem. At that time it was known that the unsolvability would follow from the existence of a single Diophantine equation that satisfied a condition that had been formulated by Julia Robinson. However, it seemed extraordinarily difficult to produce such an equation, and indeed, the prevailing opinion was that one was unlikely to exist. In my lectures, I would emphasize the important consequences that would follow from either a proof or a disproof of the existence of such an equation. Inevitably during the question period I would be asked for my own opinion as to how matters would turn out, and I had my reply ready: "I think that Julia Robinson's hypothesis is true, and it will be proved by a clever young Russian."

This book was written by that Russian. In 1970, Yuri Matiyasevich presented his beautiful and elegant construction of a Diophantine equation that satisfies Julia Robinson's hypothesis. This showed not only that Hilbert's Tenth Problem is unsolvable, but also that two fundamental concepts arising in different areas of mathematics are equivalent. The notion of recursively enumerable or semidecidable set of natural numbers from computability theory turns out to be equivalent to the purely number-theoretic notion of Diophantine set. Dr. Matiyasevich has taken full advantage of the rich interplay between the methods of elementary number theory and computability theory that this equivalence makes possible to produce a remarkable and appealing book. The reader will find new and simplified proofs of some of the main results, various extensions and applications, and many interesting exercises.

The history of the subject is recounted with meticulous care in the "Commentaries" that follow each chapter of the book. Dr. Matiyasevich has also provided a very personal account of his involvement with Hilbert's Tenth Problem in his article "My Collaboration with Julia Robinson" in the Mathematical Intelligencer (Matiyasevich [1992]). In this brief introduction, I would like to offer a few vignettes from my own involvement with the problem. As a graduate student at Princeton University, I had chosen what I knew was an excellent topic for my dissertation: the extension of Kleene's arithmetic hierarchy into the constructive transfinite, what has come to be called the hyperarithmetic hierarchy. This was a completely unex-
explored area, was quite fascinating, and was sure to yield results. But, I couldn't stop myself from thinking about Hilbert's Tenth Problem. I thought it unlikely that I would get anywhere on such a difficult problem and tried without success to discipline myself to stay away from it. In the end, my dissertation, written under the supervision of Alonzo Church, had results on both the hyperarithmetic hierarchy and Hilbert's Tenth Problem. In my dissertation, I conjectured the equivalence of the two notions mentioned above (in this book referred to as my "daring hypothesis") and saw how to improve Gödel's use of the Chinese Remainder Theorem as a coding device so as to obtain a representation for recursively enumerable sets that formally speaking seemed close to the desired result. The obstacle that remained in this so-called Davis normal form was a single bounded universal quantifier.

I met Julia Robinson at the 1950 International Congress of Mathematicians in Cambridge, Massachusetts, immediately after completing my doctorate. She had approached Hilbert's Tenth Problem from a direction opposite to mine. Where I had tried to simplify the arithmetic representation of arbitrary recursively enumerable sets, she had been trying to produce Diophantine definitions for various specific sets and especially for the exponential function. She had introduced what was to become her famous "hypothesis" and shown that under that assumption the exponential function is in fact Diophantine. It's been said that I told her that I doubted that her approach would get very far, surely one of the more foolish statements I've made in my life.

During the summer of 1957, there was an intensive five week "Institute for Logic" at Cornell University attended by almost all American logicians. Hilary Putnam and I together with our families were sharing a house in Ithaca, and he and I began collaborating, almost without thinking about it. Hilary proposed the idea of using the Chinese Remainder Theorem coding one more time to code the sequences whose existence was asserted by the bounded universal quantifier in the Davis normal form. My first reaction was skeptical. But, as pointed out in the Commentary to Chapter 3 in this book, the Chinese Remainder Theorem provides a "unique opportunity" because of the fact that polynomials preserve congruences. In fact, we were able to obtain two particular sets with quite simple definitions concerning which we were able to show that their being Diophantine would imply the same for all recursively enumerable sets.

Hilary and I resolved to seek other opportunities to work together, and we were able to obtain support for our research during the three summers of 1958, 1959, and 1960. We had a wonderful time. We talked constantly about everything under the sun. Hilary gave me a quick course in classical European philosophy, and I
gave him one in functional analysis. We talked about Freudian psychology, about
the current political situation, about the foundations of quantum mechanics, but
mainly we talked mathematics. It was during the summer of 1959 that we did our
main work together on Hilbert’s Tenth Problem. In a recent letter, Hilary wrote:

What I remember from that summer is not so much the mathematical
details as the sheer intensity with which we worked. I have never in
my life been so absorbed in a mathematical problem, and I’m sure the
same was true of you. Our method, as I remember it, was that one
of us would propose an attack and we would both work on it together,
writing on the board and arguing with each other, making suggestions,
etc., until something came of it or we reached a dead end. I could not let
go of the problem even at night; this is the only time when I regularly
stayed up to four in the morning ... I think we felt in our bones that
the problem would yield to our approach; otherwise I can’t explain the
sense of mounting excitement.

Our “approach” was still to apply the Chinese Remainder Theorem to Davis
normal form. But this time, we were combining this attack with Julia Robinson’s
methods, attempting to see if by permitting exponentiation in our Diophantine
definitions we could eliminate the troublesome bounded universal quantifier. The
problem in using the Chinese Remainder Theorem was the need for suitable moduli,
relatively prime in pairs. Gödel’s method was to obtain such moduli in an arith­
metic progression, and hence definable in Diophantine terms. We found ourselves
with the need to find exponential Diophantine definitions for sums of the reciprocals of the terms of a finite arithmetic progression as well as of the product of such
terms. To deal with the second problem, we used binomial coefficients with rational
numerators, for which we could find exponential Diophantine definitions extending
Julia Robinson’s methods, but requiring the binomial theorem with rational exponents, an infinite power series expansion. For the first, we used a rather elaborate
(and as it turned out, quite unnecessary) bit of elementary analysis, involving the
Taylor expansion of the Gamma function. Even with all that, we still couldn’t get
the full result we wanted. We needed to be able to assert that if one of our mod­
uli was a divisor of a product that it had to necessarily divide one of the factors.
And this seemed to require that the moduli be not only relatively prime in pairs,
but actual prime numbers. In the end, we were forced to assume the hypothesis
(still unproved to this date) that there are arbitrarily long arithmetic progressions
of prime numbers, in order to prove that every recursively enumerable set has an exponential Diophantine definition.

We sent our results to Julia Robinson, and she responded shortly thereafter saying:

I am very pleased, surprised, and impressed with your results on Hilbert’s Tenth Problem. Quite frankly, I did not think your methods could be pushed further . . .

I believe I have succeeded in eliminating the need for [the assumption about primes in arithmetic progression] by extending and modifying your proof. I have this written out for my own satisfaction but it is not yet in shape for anyone else.

The letter also showed quite neatly how to dispense with the messy analysis involving the Gamma function that Hilary and I had used. Soon afterwards, we received the details of Julia’s proof, and it was our turn to be “very pleased, surprised, and impressed.” She had avoided our hypothesis about primes in arithmetic progression in an elaborate and very clever argument by making use of the prime number theorem for arithmetic progressions to obtain enough primes to permit the proof to go through. She graciously accepted our proposal that our work (which had already been submitted for publication) be withdrawn in favor of a joint publication. Soon afterwards, she succeeded in a drastic simplification of the proof: where Hilary and I were trying to use the Gödel coding to obtain a logical equivalence, her elegant argument made use of the fact that the primes were only needed for the implication in one direction, and that in that direction one could make do with a prime divisor of each modulus. (Later Yuri Matiyasevich showed that in fact any sufficiently large coprime moduli could be used so that our efforts in connection with prime factors were really unnecessary; see Exercise 2 in Chapter 6.)

With the result that every recursively enumerable set has an exponential Diophantine definition combined with Julia Robinson’s earlier work on Diophantine definitions of the exponential function, it was now clear that my “daring hypothesis” of the equivalence of the two notions, recursively enumerable set and Diophantine set, was entirely equivalent to the much weaker hypothesis (now called JR) that Julia Robinson had proposed ten years earlier that one single Diophantine equation could be found whose solutions satisfied a simple condition. During the summer of 1960, Hilary and I were in Boulder, Colorado participating in a special institute intended to teach mathematicians something about physics. Hilary and I continued
to argue about quantum mechanics and explored the possibility of finding a third
degree equation to satisfy Julia Robinson's condition. It turned out once again that
we needed information that the number theorists were unable to provide, this time
about the units in pure cubic extensions of the rational numbers.

During the following years, I continued trying to prove Julia Robinson's hypoth­
thesis. I was particularly interested in trying to use what was known about quadratic
number fields. It was this work that led me to the equation \(9(x^2 + 7y^2)^2 - 7(u^2 +
7v^2)^2\), in which there is still some interest. (See the Commentary to Chapter 2.) At
this time, Julia had become rather pessimistic about JR and, for a brief period, she
actually worked towards a positive solution of Hilbert's Tenth Problem. A letter
from her dated April 1968 responding to my report on the above equation said:

\[
\text{I have enjoyed studying it, but my faith in JR still hasn't been restored.}
\]

\[
\text{However, for the first time. I can see how it might be proved. Indeed,}
\]

\[
\text{maybe your equation works, but it seems to need an infinite amount of}
\]

\[
\text{good luck!}
\]

Early in 1970, a telephone call from my old friend Jack Schwartz informed me that
the "clever young Russian" I had predicted had actually appeared. Julia Robinson
sent me a copy of John McCarthy's notes on a talk that Grigorii Tseitin had given
in Novosibirsk on the proof by the twenty-two-year-old Yuri Matiyasevich of the
Julia Robinson hypothesis. Although the notes were brief, everything important
was there, and I was able to have the great pleasure of reconstructing the proof.
But I was not satisfied until I had produced my own variant of Dr. Matiyasevich's
proof and presented it (on March 10) at a seminar at Rockefeller University at Hao
Wang's invitation.

I met Yuri a few months later at the International Congress of Mathematicians
in Nice, where he was an invited speaker. I was finally able to tell him that I had
been predicting his appearance for some time.

Martin Davis
Preface to the English Edition

I am happy to see the English translation of this book being published in the same year as the Russian original. This became possible thanks to the initiative of Albert Meyer, who proposed the publication of the translation at a time when I had not finished rewriting the Russian original for the \((n+1)\)st time. I am very grateful to him for this.

The Russian original was prepared with the aid of ChiWriter, a system that is friendly both to mathematics and to Cyrillic text. However, I continued to write the book in the old style, i.e., with pen and paper, and my wife Nina had the patience to type at least half of the book.

When the time came to prepare the first draft translation in \TeX, it turned out that the ChiWriter-to-\TeX converter did not understand many complicated formulas, and it was the skill and experience of David Jones that came to help. He and Martin Davis, one of the pioneers of Hilbert’s Tenth Problem, did a great job trying to convey into English all the nuances of Russian phrases. They also revealed many misprints and errors but of course the responsibility for the (surely) still-remaining ones is entirely mine. I express my gratitude to both of them.

Yuri Matiyasevich
In 1900 David Hilbert [1900] delivered his famous lecture entitled "Mathematische Probleme" before the Second International Congress of Mathematicians. This paper contains 23 problems, or, more precisely, 23 groups of related problems, that the nineteenth century left for the twentieth century to solve. Problem number ten is about Diophantine equations:

10. DETERMINATION OF THE SOLVABILITY OF A DIOPHANTINE EQUATION
Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: \textit{To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.}

Today we read the words "devise a process" to mean "find an algorithm." When Hilbert's Problems were posed, there was no mathematically rigorous general notion of algorithm available. The lack of such a notion was not in itself an obstacle to a positive solution of Hilbert's Tenth Problem, because for any particular algorithm it was always clear that it actually gave the desired general method for solving the corresponding problem.

During the 1930's, Kurt Gödel, Alonzo Church, Alan Turing, and other logicians provided a rigorous formulation of the notion of \textit{computability}; this made it possible to establish \textit{algorithmic unsolvability}, i.e., the impossibility of the existence of an algorithm with certain properties. Soon afterwards the first examples of algorithmically unsolvable problems were found, first in mathematical logic itself and then in other branches of mathematics.

Computability theory produced all the necessary tools for tackling the unsolvability of Hilbert's Tenth Problem. The first in a series of papers in this direction appeared at the beginning of the 1950's. The continuing effort culminated in a "negative solution" of Hilbert's Tenth Problem in 1970.

As in the case of many other problems whose solution was long awaited, the technique developed for the resolution of Hilbert's Tenth Problem is of independent value because it has other applications. Some of these are rather striking, and taken together, these other applications are perhaps even more important than the solution of Hilbert's Tenth Problem. The main technical result implying the unsolvability of Hilbert's Tenth Problem asserts that the class of Diophantine sets is identical with the class of \textit{recursively enumerable sets}. Another corollary of this same result that uses no special terminology states: \textit{It is possible to exhibit}
explicitly a polynomial with integer coefficients such that the set of all the positive values it assumes for integer values of its variables is exactly the set of all prime numbers.

The present book is devoted to the algorithmic unsolvability of Hilbert’s Tenth Problem and related topics; the numerous partial results that have been obtained in the direction of a positive solution of the Problem are hardly considered.


A distinguishing feature of this book is that, in addition to presenting the negative solution of the problem, it contains a number of diverse applications of the technique developed for that solution. At present these applications are scattered among various publications, mainly journal articles. During the twenty years that have passed since the problem was “unsolved,” many improvements and modifications of the original proof have been suggested. In addition to these, the book contains several new, previously unpublished proofs.

Understanding the negative solution of Hilbert’s Tenth Problem naturally requires some knowledge from both number theory and mathematical logic. Wishing to make the book accessible to a broader readership, especially to younger mathematicians, the author has tried to reduce the mathematical prerequisites needed by the reader. In particular, no knowledge of computability theory is presupposed. All the necessary notions are defined in the book, and thus it can serve as an introduction to this fascinating subject (but of course the book cannot be used for its systematic study). A few number-theoretical results that are usually not part of the basic mathematical curriculum are proved in the Appendix.

Each chapter is concluded by Exercises, which contain problems of varying difficulty. While some of them are quite elementary, others amount to small research problems. The aim of the exercises is to present diverse results, but without proofs. Of course, this division (on the one hand, the main content of the book presented with full proofs and, on the other, the results relegated to the exercises) represents a rather subjective judgement by the author. In particular, the exercises contain results requiring special knowledge or having cumbersome proofs as well as results that are, most likely, far from the best possible or that were thought to be of lim-
ited interest. The *Hints* supply the ideas of the proofs and/or references to the literature.

In addition to the exercises, there are a few *Open Questions* and *Unsolved Problems*. Again, the division is quite subjective. It may be that an open question has not been settled simply because no one has tackled it seriously, and the answer may turn out to be without significance. On the other hand, the unsolved problems have attracted the attention of many capable researchers, and solving these problems may require decades.

Each chapter is completed by a *Commentary* providing a historical view of its contents. This seems to be worthwhile because the logical order of presentation used in the book often doesn’t coincide with the chronological order in which the results had originally been obtained.

The *Bibliography* lists all of the principal publications concerned with the negative solution to Hilbert’s Tenth Problem as well as most of the papers that apply the technique developed for obtaining that solution. The author would be grateful if omissions were called to his attention.

The book need not be read consecutively. It may be said to consist of two parts. The first part, consisting of Chapters 1–5, presents the solution of Hilbert’s Tenth Problem. Figure 1 exhibits the dependencies among the sections of the first part. To determine which sections should be read before a particular section U.V, locate that section in the figure and imagine vertical and horizontal coordinate axes placed on the page in such a manner that their point of intersection is at the period in U.V. Then, before reading section U.V you will need to read all sections lying in the upper left quadrant including its boundary.

Similarly, Figure 2 shows how the sections of the second part, devoted to applications, depend on the sections from the first part (but not dependencies between sections of the first part and not dependencies between sections of the second part):

There are few dependencies among the sections of the second part. Sections 6.1–6.3 offer three different techniques for achieving the same aim; it suffices to know any one of them in order to read Sections 6.4–6.6 and 9.1. Similarly, Sections 4.5 and 6.5 present two different constructions of a universal equation, either of which can serve as background for reading Sections 6.6 and 8.1. Section 8.2 presupposes acquaintance with Section 7.2, which in turn presupposes knowledge of Sections 6.2–6.3; Section 9.4 uses results from Section 9.2, and Section 10.1 is based on Section 6.6.